Universität Konstanz
Fachbereich Mathematik und Statistik
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Exercises for the course "Linear Algebra I"

Sheet 9

Hand in your solutions on Thursday, 09. Januar 2020, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 9.1 ***Beweismechanikaufgabe*** *(4 points)*

Bitte gehen Sie in dieser Aufgabe nach den Regeln der Beweismechanik vor und geben Ihre Lösung auf einem separaten Blatt in den Briefkasten mit der Aufschrift „Beweismechanikaufgaben" ab. Ihnen unbekannte Begriffe und Symbole können Sie in der Beweismechanik nachschlagen.

Sei V ein $\mathbb{R}$-Vektorraum und sei

$$\mathcal{L} := \{f : f : \mathbb{R}^3 \to V \text{ ist linear}\}.$$

(i) Es gilt folgende Aussage (die Sie nicht beweisen müssen):
$\mathcal{L}$ ist bezüglich punktweiser Addition und Skalarmultiplikation ein $\mathbb{R}$-Vektorraum.
Erklären Sie (im Stile einer Definition), was hierbei mit „punktweiser Addition und Skalarmultiplikation" gemeint ist.

(ii) Seien nun $v, w \in V \setminus \{0\}$ und seien $f : \mathbb{R}^3 \to V$, $x \mapsto (x_1 + x_2)v$ und $g : \mathbb{R}^3 \to V$, $x \mapsto (x_1 - x_2)w$. Zeigen Sie, dass $f, g \in \mathcal{L}$ gilt und dass f und g linear unabhängig sind.

Definition for Exercise 9.2: Let M, N be sets and $f : M \to N$ a mapping. Then f is called

(1) *injective* if and only if $\forall x, y \in M\colon f(x) = f(y) \Rightarrow x = y$.

(2) *surjective* if and only if $f(M) = N$, i.e. if and only if $\forall y \in N : \exists x \in M : f(x) = y$.

(3) *bijective* if and only if f is both injective and surjective.

Exercise 9.2 *(4 points)*

Let K be a field, V and W finite dimensional K-vector spaces and $f : V \to W$ a linear mapping. Show that:

(a) f is injective if and only if for all $v_1, \ldots, v_n \in V$ the following holds: if $\{v_1, \ldots, v_n\}$ is linearly independent, then so is $\{f(v_1), \ldots, f(v_n)\}$.

(b) f is surjective if and only if for all $v_1, \ldots, v_n \in V$ the following holds: if $V = \operatorname{span}\{v_1, \ldots, v_n\}$, then also $W = \operatorname{span}\{f(v_1), \ldots, f(v_n)\}$.

(c) f is an isomorphism if and only if for all $v_1, \ldots, v_n \in V$ the following holds: If $\{v_1, \ldots, v_n\}$ is a basis of V, then $\{f(v_1), \ldots, f(v_n)\}$ is a basis of W.

(d) f is an isomorphism if and only if f^{-1} is an isomorphism.

Exercise 9.3 *(4 points)*

In Corollary 16.1 it was proven that an $(n \times n)$-matrix A is invertible, if it row vectors are linearly independent. Show the converse: if an $(n \times n)$-matrix is invertible, then the row vectors of A are linearly independent.

Exercise 9.4 *(4 points)*

Let $+$ and $\cdot$ denote the usual addition and multiplication of the real numbers. Then $(\mathbb{R}, +, \cdot)$ is a $\mathbb{Q}$-vector space. In the following, you may use this fact without proving it.

(a) Show the following: If $X \subseteq \mathbb{R}$ is linearly dependent over $\mathbb{Q}$, then there exist $n \in \mathbb{N}$, $x_1, ..., x_n \in X$ and $z_1, ..., z_n \in \mathbb{Z}$ such that $\Sigma_{i=1}^{n} z_i x_i = 0$ and $z_i \neq 0$ for some $i \in \{1, \ldots, n\}$.

Now let $\mathbb{P} \subseteq \mathbb{N}$ denote the set of all prime numbers. If x is a positive real number, we denote by $\mathrm{ld}(x)$ the unique real number y such that $2^y = x$. In the following we assume without proof that such a unique y exists to every positive real number x. The element $\mathrm{ld}(x)$ is called the *dual logarithm* of x.

(b) Show the following: The set $\{\mathrm{ld}(p) : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Deduce that $\mathbb{R}$, considered as a $\mathbb{Q}$-vector space, admits no finite basis.

Remark: This does not imply yet that $\mathbb{R}$ has infinite dimension over $\mathbb{Q}$. To this end, it remains to show that $\mathbb{R}$ even admits a basis as a $\mathbb{Q}$-vector space. However, in the lecture Linear Algebra 2, we will see that any vector space admits a basis.

Bonus Excercise 9.5 *(4 Extra Points)*

Decide, whether the following sets are linearly dependent or linearly independent over the respective vector spaces and fields.

(a) $\{1, \sqrt{2}, \sqrt{3}\} \subseteq \mathbb{R}$ over $\mathbb{R}$.

(b) $\{1, \sqrt{2}, \sqrt{3}\} \subseteq \mathbb{R}$ over $\mathbb{Q}$.

(c) $\{\left(123, \frac{3}{4}\right), \left(\cos\frac{1}{3}, \sin\frac{7}{13}\right), \left(\sqrt{17}, -\frac{50}{49}\right), \left(\pi, \frac{3\pi}{177}\right)\} \subseteq \mathbb{R}^2$ over $\mathbb{R}$

(d) $\{2, 1+2X, X^2, 2X^2+X^3, X+2X^3\} \subseteq \mathbb{F}_3[X]$, where $\mathbb{F}_3[X]$ denotes the set of all polynomials in X with coefficients in $\mathbb{F}_3$ (cf. Excercise 8.1).

Bonus Excercise 9.6 *(3 Extra Points)*

Let $a, b \in \mathbb{N}_0$ such that $\gcd(a, b) = 1$ and let $n \in \mathbb{N}$ such that $n \geq ab$. Show that there exist $x, y \in \mathbb{N}_0$ such that $ax + by = n$.